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# Generalized Boundary Integral Equation for Transient Heat Conduction in Heterogeneous Media

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**Using the generalized boundary integral method developed by the authors for steady heat conduction in heterogeneous media as a point of departure, a generalized dual reciprocity boundary element method (BEM) is presented for the solution of transient heat conduction problems in heterogeneous media. In the process, new interpolating radial basis functions are defined. This method retains the boundary-only discretization feature of the BEM. Two- and three-dimensional numerical examples provide validation of the proposed method. Excellent agreement is found between analytical and BEM-computed results.**

## Nomenclature

| | |
|---|---|
| $A$ | = amplification factor |
| $c$ | = specific heat |
| $D$ | = generalized forcing function |
| $E$ | = generalized fundamental solution |
| $f_k(x)$ | = dual reciprocity expansion function, $f$ |
| $G_{i,j}$, $H_{i,j}$ | = influence coefficients |
| $k$ | = thermal conductivity |
| $\hat{n}$ | = outward-drawn normal |
| $p_k(x)$ | = normal derivative of the dual reciprocity expansion function $u$ |
| $q$ | = heat flux |
| $q_j$ | = negative of conductive heat flux, $(k\partial T/\partial n)_j$ |
| $r$ | = radial direction both in polar and spherical geometries |
| $r_k$ | = radial distance from $k$th dual reciprocity expansion point |
| $r$, $\theta$ | = polar coordinate system |
| $r$, $\theta$, $\phi$ | = spherical coordinate system |
| $T$ | = temperature |
| $t$ | = time |
| $u_k$ | = dual reciprocity expansion function, $u$ |
| $x$ | = generalized coordinate |
| $x_i$ | = $x$ location of the source point |
| $x$, $y$, $z$ | = coordinates of Cartesian system |
| $y_i$ | = $y$ location of the source point |
| $\alpha_k$ | = expansion coefficient |
| $\Gamma$ | = domain boundary |
| $\delta$ | = Dirac delta function |
| $\theta$ | = polar angle (polar geometry), azimuthal angle (spherical geometry) |
| $\theta_t$, $\theta_q$ | = Newmark parameters |
| $\xi$ | = location of source point |
| $\rho$ | = density |
| $\phi$ | = zenith angle in spherical geometry |
| $\Omega$ | = domain of the problem |
| $\nabla$ | = gradient operator |

## Introduction

MODERN industrial materials, for instance, functionally gradient materials,[1] exhibit thermophysical property heterogeneities that can be tailored by careful design of their microstructure[1,2] to meet ever-increasing demands of emerging technologies such as the single-stage-to-orbit plane, ceramic engines, and the advanced turbine system initiative. Heat conduction analysis in heterogeneous media has found renewed importance in engineering practice. However, the governing equation for heat conduction in heterogeneous media is a variable coefficient partial differential equation (PDE). The very nature of this equation often precludes analytical solution even in simple geometries. Consequently, traditional numerical techniques such as the finite difference method, finite volume method, or finite element method have been successfully applied to obtain numerical solutions. Another powerful numerical technique that has been used to solve heat conduction problems is the integral equation-based boundary element method (BEM) detailed in Brebbia et al.[3] and Banerjee.[4]

One of the distinct features of the BEM is that, for many field problems of engineering, a boundary integral equation (BIE) is discretized to solve the field problem of interest. Consequently, only the bounding surface(s) of the domain is discretized. However, in certain cases, such as nonlinear heat conduction and elastostatics, the BEM leads to domain integrals that detracts from the boundary-only feature of the method. Generally, a BEM formulation of heat conduction in heterogeneous media also leads to domain integrals. However, in certain cases, such as layered media, the conductivity can be taken as constant in certain zones of the medium. A common approach taken to address this type of heterogeneous medium is to subdivide the domain into multizones, for each of which the material property is taken to be constant.[5–7] In this approach, boundary integrals are written for each zone and these are coupled by enforcing interfacial continuity of temperature and heat flux. This leads to banded block matrix equations for which efficient solvers have been specially tailored.[8] Another approach, taken by Lafe and Cheng,[9] to solve potential problems in inhomogeneous media is to use the free-space Green's function for the Laplace equation and recast the governing equation into a Poisson equation with a spatially varying forcing term and, subsequently, develop a perturbation method capitalizing on small differences between the homogeneous and heterogeneous case to eliminate the domain integral. A boundary-only formulation for certain heat conduction problems with limited one-dimensional functional variations in thermal conductivity variation was also developed.[10–12] A new free-space Green's function has been discovered for potential problems in two- and three-dimensional heterogeneous media that are, however, restricted to one-dimensional linear variations with position.

In this paper, we present a general technique for the development of a BIE for steady and transient heat conduction in

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heterogeneous media. Our approach is based on the definition of a new singular generalized nonsymmetric forcing function, whose sampling properties are tailored to permit the development of a generalized BIE for the governing variable coefficient PDE. This integral equation provides an expression for the temperature in terms of contour integrals only. First, the steady-state integral equation developed by Kassab and Divo[16,17] for heat conduction in heterogeneous media is presented. Recent extension of the approach to steady-state formulation for heterogeneous anisotropic media can be found in Divo and Kassab,[18,19] where detailed derivations and a collection of examples can be found. Using this as a point of departure, a dual reciprocity transient BEM is then derived for the diffusion equation. In the process, new radial interpolating basis functions are defined for heterogeneous media.[20] Numerical implementation is discussed. Quadratic isoparametric elements are used in two dimensions, and bilinear isoparametric elements are used in three dimensions. Numerical examples are provided to validate the proposed theoretical developments. Several arbitrary spatial variations of thermal conductivity are considered. Numerical results are compared to analytical solutions.

## Development of Generalized BIE for Steady-State Heat Conduction in Heterogeneous Media

To develop a transient boundary integral formulation for heat conduction in heterogeneous media it is necessary to briefly review the generalized BIE recently developed by the authors for steady-state heat conduction in heterogeneous media. The governing equation for steady-state heat conduction in isotropic heterogeneous media is

$$\nabla \cdot [k(x)\nabla T(x)] = 0 \tag{1}$$

where $k(x)$ is the spatially varying thermal conductivity, and $T(x)$ is the temperature. The preceding variable coefficient PDE is now converted to an integral equation. Following standard integral equation methods, a function $E(x, \xi)$ is introduced and integrated into the product of the governing equation and $E(x, \xi)$ over the domain $\Omega$ of the problem

$$\int_\Omega \{E(x, \xi)\nabla \cdot [k(x)\nabla T(x)]\}\, d\Omega = 0 \tag{2}$$

The domain $\Omega$ can be one-, two-, or three-dimensional. Using Green's first identity twice, the following integral equation is derived:

$$\oint_\Gamma \left[E(x, \xi)k(x)\frac{\partial T}{\partial n}(n) - T(x)k(x)\frac{\partial E}{\partial n}(x, \xi)\right] d\Gamma(x) + \int_\Omega \{T(x)\nabla \cdot [k(x)\nabla E(x, \xi)]\}\, d\Omega = 0 \tag{3}$$

Here, the domain boundary is denoted by $\Gamma$. The dimension of $\Gamma$ is dictated by the dimension of $\Omega$. The solution $E$ to the adjoint operator perturbed by a singular forcing function $D$ acting at the source point $\xi$ is defined by

$$\nabla \cdot [k(x)\nabla E(x, \xi)] = -D(x, \xi) \tag{4}$$

For those problems in which the thermal conductivity does not vary with position, $D(x, \xi)$ is traditionally taken as the Dirac delta function. In this case, the solution of Eq. (4) in an infinite domain is readily derived as the well-known fundamental (or Green's free-space) solution to the steady-state diffusion equation, $-(1/2\pi k)\ell n(r)$. However, the fundamental solution for arbitrary heterogeneous heat conduction cannot be obtained because the traditional attempts at solving the problem relied on the Dirac delta function as a forcing function in Eq. (4). Because the Dirac delta function is symmetric about its source point $\xi$ and the adjoint equation is a variable coefficient PDE, the fundamental solution must be nonsymmetric.

To overcome this difficulty, Kassab and Divo[16,17] introduced a generalized forcing function, $D(x, \xi)$, which obeys the following properties:

$$\nabla \cdot [k(x)\nabla E(x, \xi)] = -D(x, \xi) \tag{5a}$$

$$\int_{\Omega_{,c}} D(x, \xi)\, d\Omega(x) = 1 \tag{5b}$$

$$\int_\Omega T(x)D(x, \xi)\, d\Omega(x) = T(\xi)A(\xi) + \varepsilon(\xi) \tag{5c}$$

$$A(\xi) = \int_\Omega D(x, \xi)\, d\Omega(x) \tag{5d}$$

In Eq. (5), the domain $\Omega_{,c}$ is a circular domain centered about the point $\xi$, whereas the domain $\Omega$ is arbitrary in shape. By constructing such a set of relations, we will generate a forcing function $D$. It will be shown later that $D$ is a singular nonsymmetric forcing function at the source point $\xi$, which is actually composed of the sum of a Dirac delta function $\delta$ and a nonsymmetric dipole-like function $D_d$. The sampling properties of $D$ are analogous but markedly different from those of the Dirac delta function as evidenced by Eq. (5). It is noted that in early papers by Divo and Kassab,[16–19] only the $D_d$ portion of the $D$ function was plotted and reported, and that the $\varepsilon(\xi)$ term as given in Eq. (5c) was neglected (Fig. 1). Divo and Kassab[21] discuss and incorporate both of these revisions. The consequences of the preceding definitions for $D$ permit us to seek a locally symmetric solution, $E(x, \xi)$, to the adjoint equation and, thus, derive a BIE. We refer to $A(\xi)$ as the amplification factor. Introducing Eq. (5a) into (5d) and applying the Gauss divergence theorem, $A(\xi)$ is evaluated by a contour integral as

$$A(\xi) = -\oint_\Gamma \left[k(x)\frac{\partial E}{\partial n}(x)\right] d\Gamma(x) \tag{6}$$

The amplification factor explicitly depends on the solution of the adjoint equation $E$ and the thermal conductivity. Further, it is noted that if the thermal conductivity is constant, the amplification factor reduces to the well-known results of $A = 1$ for any interior point and $A = 1/2$ for any point on a smooth boundary.

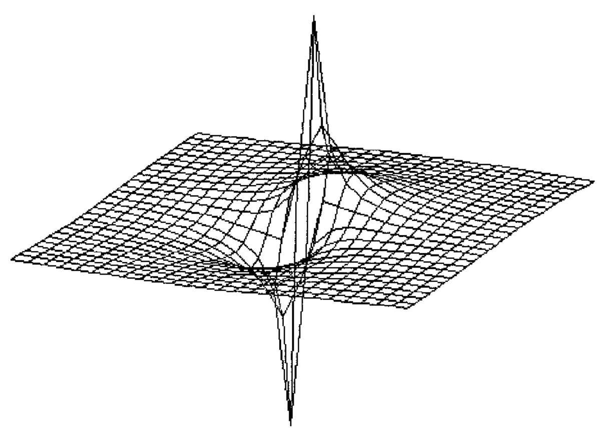

**Fig. 1 Plot of dipole-like $D_d$ component of $D$ in proximity of $\xi$.**

Finally, invoking the sampling property of $D$, and upon substitution into Eq. (3), the desired BIE for the temperature is obtained as

$$A(\xi)T(\xi) + \varepsilon(\xi) = \oint_\Gamma \left[E(x, \xi)k(x)\frac{\partial T}{\partial n}(x)\right] d\Gamma(x) - \oint_\Gamma \left[T(x)k(x)\frac{\partial E}{\partial n}(x, \xi)\right] d\Gamma(x) \tag{7}$$

The amplification factor $A(\xi)$ must be evaluated at all points where the temperature is sought, whether on the boundary $\Gamma$ or the interior of the finite domain $\Omega$. This integral equation can readily be solved by standard BEM techniques once the function $E(x, \xi)$ is determined. The details of the derivation of the generalized fundamental solution can be found in Kassab and Divo[16–19] and will now be briefly reviewed for two-dimensional problems. It is convenient to consider polar coordinates in seeking the fundamental solution. In a polar coordinate system $r - \theta$ centered about the given location of the source point $\xi = (x_i, y_i)$, the adjoint equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rk(r, \theta, x_i, y_i)\frac{\partial E}{\partial r}\right] + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left[k(r, \theta, x, y_i)\frac{\partial E}{\partial \theta}\right] = -D(\xi, x) \tag{8}$$

The thermal conductivity $k$ depends explicitly on the location of the source point because of the local polar coordinate system used in this derivation. If we seek a solution to Eq. (8) that is locally symmetric, that is, $E = E(r, x_i, y_i)$, then the generalized singular forcing function $D$ must be nonsymmetric, i.e., $D(r, \theta, x_i, y_i)$. In this case, the adjoint equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left[rk(r, \theta, x_i, y_i)\frac{\partial E(r, x_i, y_i)}{\partial r}\right] = -D(r, \theta, x_i, y_i) \tag{9}$$

To solve Eq. (9), Eq. (5) is used and a first integration leads to[16]

$$\frac{\partial E}{\partial r}(r, x_i, y_i) = \frac{g(r, x_i, y_i)}{r} \tag{10}$$

A second integration leads to the form of the generalized fundamental solution, $E(r, x_i, y_i)$, as

$$E(r, x_i, y_i) = \int \frac{g(r, x_i, y_i)}{r}\, dr \tag{11}$$

To fix the value of $g(r, x_i, y_i)$, we integrate Eq. (5a) over a circular domain $\Omega_{,c}$ centered about the source point $\xi = (x_i, y_i)$

$$\int_{\Omega,c} \nabla\cdot[k(r, \theta, x_i, y_i)\nabla E(r, x_i, y_i)]\, d\Omega_{,c} = -\int_{\Omega,c} D(x, \xi)\, d\Omega_{,c} \tag{12}$$

The right-hand side (RHS) integral is unity according to property (5b) of $D$, and applying the Gauss divergence theorem to the left-hand side results in

$$\int_0^{2\pi}\left[rk(r, \theta, x_i, y_i)\frac{\partial E}{\partial r}(r, x_i, y_i)\right] d\theta = -1 \tag{13}$$

Substituting from Eq. (10) for the term in the square bracket, we arrive at

$$\int_0^{2\pi}[g(r, x_i, y_i)k(r, \theta, x_i, y_i)]\, d\theta = -1 \tag{14}$$

or, we now have the desired result for the function $g(r)$ as

$$g(r, x_i, y_i) = \frac{-1}{\displaystyle\int_0^{2\pi} k(r, \theta, x_i, y_i)\, d\theta} \tag{15}$$

and the expression for the generalized fundamental solution $E(r, x_i, y_i)$ is then

$$E(r, x_i, y_i) = -\int \frac{dr}{r\displaystyle\int_0^{2\pi} k(r, \theta, x_i, y_i)\, d\theta} \tag{16}$$

This expression can now be used for arbitrary spatial variations of the thermal conductivity. The generalized fundamental solution is locally symmetric about the source point $(x_i, y_i)$, but varies from point to point in the domain as the source point is moved. In the case of constant thermal conductivity Eq. (16) yields the familiar result for the two-dimensional Laplace equation, $E(r) = -(1/2\pi k)\ell n\, r$. The normal derivative of the fundamental solution is readily evaluated as

$$\frac{\partial E}{\partial n} = \frac{\partial E}{\partial r} n_r = \frac{-n_r}{r\displaystyle\int_0^{2\pi} k(r, \theta, x_i, y_i)\, d\theta} \tag{17}$$

Here, $n_r$ is the radial component of the outward drawn unit normal vector. Again, it is noted that the $-1/2\pi kr$ behavior of the normal derivative of the free-space solution for the case of constant $k$ is retrieved. For the three-dimensional case, the adjoint equation is recast using a spherical coordinate system centered about the source point $(x_i, y_i, z_i)$, and again seeking a locally symmetric fundamental solution we arrive at

$$E(r, x_i, y_i, z_i) = -\int \frac{dr}{r^2\displaystyle\int_0^{2\pi}\int_0^{\pi} k(r, \theta, \phi, x_i, y_i, z_i)\sin\theta\, d\theta\, d\phi} \tag{18}$$

Again, the well-known three-dimensional fundamental solution for the heat conduction equation in homogeneous media, $1/4\pi rk$, is retrieved from Eq. (18). The preceding generalized fundamental solutions can readily be derived once the thermal conductivity is specified. For instance, given a bilinear two-dimensional variation of the thermal conductivity, $k(x, y) = a + bx + cy + dxy$, the generalized fundamental solution is readily derived as $E(r, x_i, y_i) = -\ell n(r)/[2\pi(a + bx_i + cy_i + dx_iy_i)]$. It is noted that the formation of the generalized fundamental solution proposed in this paper requires nearly the same effort in comparison to the formation of traditional fundamental solutions for constant property differential equations as long as the indefinite integral in Eqs. (16) and (18) can be evaluated analytically.

A close examination of $D(x, \xi)$ reveals that it is actually composed of two parts: 1) A Dirac delta function $\delta$ and 2) a nonsymmetric dipole-like function $D_d$. To demonstrate this consider the specific case of a two-dimensional $k(x)$ taken as

$$k(x, y) = xy \tag{19}$$

In this case the generalized fundamental solution $E(x, \xi)$ is obtained as

$$E(x, \xi) = -(1/2\pi x_iy_i)\ell n(r) \tag{20}$$

Introducing Eq. (20) in the adjoint equation, $D(x, \xi)$ is derived as

$$D(x, \xi) = \frac{1}{r}\frac{\partial}{\partial r}\left[\frac{(r\cos\theta + x_i)(r\sin\theta + y_i)}{2\pi x_i y_i}\right] \tag{21}$$

or carrying out the differentiation there results

$$D(x, \xi) = \left[\frac{y_i\cos\theta + 2r\cos\theta\sin\theta + x_i\sin\theta}{2\pi r x_i y_i}\right] + \left\{\frac{1}{2\pi r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(\ell n\ r)\right]\right\} \tag{22}$$

The first term in Eq. (22) is the nonsymmetric dipole-like term hereon referred to as $D_d(x, \xi)$, and the second term is the cylindrical polar form of the Dirac delta function $\delta(x, \xi)$.[23] The forcing function $D(x, \xi)$ is actually expressed in general as

$$D(x, \xi) = D_d(x, \xi) + \delta(x, \xi) \tag{23}$$

It is readily verified that the normalizing property in Eq. (5b) is satisfied. Integrating $D(x, \xi)$ in Eq. (22) over a circle centered at $\xi$, it is clear that the integral of $D_d(x, \xi)$ over the area of the circle vanishes and that the integral of the Dirac delta component over the circle contributes one, thus validating the normalizing property. With this fact established, the exactness of the sampling property of $D(x, \xi)$ in Eq. (5c) can now be verified

$$\begin{aligned}\int_\Omega T(x)D(x, \xi)\,d\Omega(x) &= \int_\Omega [T(x) \pm T(\xi)]D(x, \xi)\,d\Omega(x)\\ &= T(\xi)\int_\Omega D(x, \xi)\,d\Omega(x) + \int_\Omega [T(x) - T(\xi)]D(x, \xi)\,d\Omega(x)\\ &= T(\xi)A(\xi) + \varepsilon(\xi)\end{aligned} \tag{24}$$

where $A(\xi)$ is evaluated using the contour integrals given in Eq. (6). In light of the fact that $D(x, \xi) = D_d(x, \xi) + \delta(x, \xi)$, the area integral becomes

$$\varepsilon(\xi) = \int_\Omega [T(x) - T(\xi)]D_d(x, \xi)\,d\Omega(x) \tag{25}$$

which establishes the exactness of Eq. (5c). The $\varepsilon(\xi)$ term often has a negligible contribution; however, it cannot be formally neglected. Equation (25) provides a measure of the error if $\varepsilon(\xi)$ is neglected. However, it is evident that $\varepsilon(\xi)$ involves an area integral, which is an undesirable feature.

Divo and Kassab[21,22] developed a method to estimate the term $\varepsilon(\xi)$ using an expression containing only boundary integrals. The integrand in Eq. (25) is composed of the product of $D_d(x, \xi)$, which is known exactly once the thermal conductivity is specified for the problem. For example, as in Eq. (22) as discussed earlier, and $[T(x) - T(\xi)]$ that depends on the solution. Denoting the integrand at any source point $\zeta_i$ by

$$h^i(x) = [T(x) - T(\xi_i)]D_d(x, \xi_i) \tag{26}$$

this term is well-behaved as $[T(x) - T(\xi)]$ goes to zero and $D_d(x, \xi)$ becomes odd singular as $\xi$ is approached, and the product of these two vanishes in this limit. Therefore, we can expand $h^i(x)$ in a series

$$h^i(x) = \sum_{j=1}^{N} \alpha_j^i f_j(x) \tag{27}$$

To reduce the area integral to a contour integral, a new expansion is proposed in terms of functions $f_j(x)$, each satisfying $f_j(x) = \nabla\cdot\boldsymbol{u}_j(x)$. Here, we take the expansion functions $f_j(x)$ to be the $1 + r$ radial basis functions proposed by Powell[20] and used widely in the dual reciprocity boundary element method (DRBEM).[24] Here, $r$ is the radial distance between the expansion point $x_j$ and the field point $x$. In two dimensions, for instance, the antidivergence of $1 + r$ leads to the following expression for $\boldsymbol{u}_j(x)$:

$$\boldsymbol{u}_j(x) = [\tfrac{1}{3}r^2(x, x_j) + \tfrac{1}{2}r(x, x_j)]\boldsymbol{e}_r \tag{28}$$

where $\boldsymbol{e}_r$ is the unit vector in the radial direction, referenced to a local polar coordinate system fixed at $x_j$. Introducing Eqs. (26–28) into Eq. (25), then with the use of the Gauss divergence theorem, we can express $\varepsilon(\xi_i)$ in terms of contour integrals as

$$\varepsilon(\xi_i) = \sum_{j=1}^{N} \alpha_j^i \left[\oint_\Gamma \boldsymbol{u}_j(x)\cdot\hat{n}(x)\,d\Gamma(x)\right] \tag{29}$$

The contour integrals are evaluated using the same discretization employed for the BEM. To determine the coefficients $\alpha_j^i$, we collocate the expansion in Eq. (27) at $(x_k)$, $k = 1, 2 \ldots N$, points (in all our numerical examples we only use the BEM boundary points for this purpose), and in a procedure similar to DRBEM we obtain

$$h^i(x_k) = \sum_{j=1}^{N} \alpha_j^i f_j(x_k) \tag{30}$$

and solve the vector $\boldsymbol{\alpha}^i$ as

$$\boldsymbol{\alpha}^i = F^{-1}\boldsymbol{h}^i \tag{31}$$

where $F$ is an interpolating matrix and $\boldsymbol{h}^i$ is a vector whose elements are composed of $h^i(x_k)$. Equation (7) can now be expressed purely in terms of contour integrals and can be discretized following a standard BEM procedure to lead to the following set of equations:

$$A(\xi_i)T(\xi_i) + \varepsilon(\xi_i) + \sum_{j=1}^{N}\hat{H}_{ij}T_j = \sum_{j=1}^{N} G_{ij}q_j, \qquad i = 1, 2 \ldots N \tag{32}$$

where, for convenience, $q_j$ is defined as $(k\partial T/\partial n)_j$ (thus, $q$ is the negative of the conductive heat flux). Equation (32) symbolically leads to

$$\sum_{j=1}^{N} H_{ij}T_j + \varepsilon^i = \sum_{j=1}^{N} G_{ij}q_j \tag{33}$$

where the components of the influence matrix $[H]$ are $H_{ij} = \hat{H}_{ij} + \delta_{ij}A(\xi_i)$. The influence coefficients are evaluated numerically by Gauss-type quadratures. Once the boundary conditions are introduced, we retrieve the algebraic form

$$[A]\{\boldsymbol{x}\} = \{\boldsymbol{b}\} - \{\boldsymbol{\varepsilon}(T)\} \tag{34}$$

The vector $\{\boldsymbol{\varepsilon}(T)\}$ can be directly absorbed into the linear system without resorting to iteration by taking advantage of the fact that the error terms $\varepsilon(\xi_i)$ have a linear relationship with respect to the temperature field $\{\boldsymbol{T}\}$. Departing from Eq. (29) we may express

$$\varepsilon(\xi_i) = \sum_{j=1}^{N} \alpha_j^i \boldsymbol{v}_j \tag{35}$$

where the components of the vector $\boldsymbol{v}_j$ are the closed-contour integral of the antidivergence expansion functions $\boldsymbol{u}_j(x)$ dotted into the outward-drawn unit normal $\hat{n}(x)$, which are independent of the resulting temperature field because of the geometry-only dependence of the integrand. Substituting Eq. (31) for the coefficients $\boldsymbol{\alpha}^i$ into Eq. (35) leads to

$$\varepsilon^i = \boldsymbol{w}^t \boldsymbol{h}^i \tag{36}$$

where the transposed vector $\boldsymbol{w}^t = \boldsymbol{v}^t F^{-1}$ and, as previously defined, the components of the vector $\boldsymbol{h}^i$ are the integrand of the expression for the error term $\varepsilon(\zeta_i)$ evaluated at the location $x_k$ for each collocation point $\xi_i$, that is,

$$h_k^i = (T_k - T^i) D_{d_k}^i \tag{37}$$

Introducing Eq. (37) into Eq. (36), the error $\varepsilon^i$ at every collocation point $\xi_i$ is obtained as

$$\varepsilon^i = \sum_{k=1}^{N} w_k (T_k - T^i) D_{d_k}^i \tag{38}$$

Finally, Eq. (33) is rearranged as

$$\sum_{j=1}^{N} H_{ij} T_j + \sum_{k=1}^{N} w_k (T_k - T^i) D_{d_k}^i = \sum_{j=1}^{N} G_{ij} q_j \tag{39}$$

or in compact form

$$\sum_{j=1}^{N} \tilde{H}_{ij} T_j = \sum_{j=1}^{N} G_{ij} q_j \tag{40}$$

where the modified influence coefficients $\tilde{H}_{ij}$ are defined as

$$\tilde{H}_{ii} = H_{ii} - \sum_{\substack{k=1 \\ k \neq i}}^{N} w_k D_{d_k}^i, \qquad \tilde{H}_{ij} = H_{ij} + w_j D_{d_j}^i \tag{41}$$

which leads to the standard algebraic form, $[A]\{\boldsymbol{x}\} = \{\boldsymbol{b}\}$, upon introduction of the boundary conditions.

This completes the presentation of the generalized BIE for steady-state heat conduction in heterogeneous media. Using the preceding developments as a point of departure, a generalized dual reciprocity boundary element formulation is now developed for transient heat conduction in heterogeneous media.

## Generalized DRBEM for Transient Heat Conduction in Heterogeneous Media

In the BEM transient problems can be solved by developing a transient fundamental solution and a corresponding transient BIE by using Laplace transforms to convert the transient problem to a boundary value problem in Laplace space, or by using the steady-state fundamental solution and the dual reciprocity method.[24] In this paper we adopt the latter method and generalize it to address transient heat conduction in heterogeneous media. The governing equation of interest is

$$\nabla \cdot [k(x) \nabla T(x, t)] = \rho c \frac{\partial T}{\partial t} \tag{42}$$

In the dual reciprocity method, the RHS of Eq. (42) is expanded in series as

$$\rho c \frac{\partial T}{\partial t} = \sum_{k=1}^{N+L} \boldsymbol{\alpha}_k \nabla \cdot [k(x) \nabla u_k] \tag{43}$$

where $N$ is the number of boundary nodes, $L$ is the number of internal dual reciprocity points, and $u_k$ are functions to be defined in the following text. Substitution of Eq. (43) into Eq. (42) leads to

$$\nabla \cdot [k(x) \nabla T(x, t)] = \sum_{k=1}^{N+L} \boldsymbol{\alpha}_k \nabla \cdot [k(x) \nabla u_k] \tag{44}$$

Multiplying both sides by $E$ and integrating over the domain of interest

$$\int_\Omega E(x, \xi) \nabla \cdot [k(x) \nabla T(x)] \, d\Omega = \sum_{k=1}^{N+L} \boldsymbol{\alpha}_k \int_\Omega E(x, \xi) \nabla \cdot [k(x) \nabla u_k] \, d\Omega \tag{45}$$

Applying Green's first identity twice in both sides of Eq. (45)

$$\begin{aligned} A(\xi) T(\xi) + \varepsilon(\xi) + \oint_\Gamma [q(x) E(x, \xi) - F(x, \xi) T(x)] \, d\Gamma \\ = \sum_{k=1}^{N+L} \boldsymbol{\alpha}_k \{ A(\xi) u_k(\xi) + \varepsilon_u(\xi) \\ + \oint_\Gamma [p_k(x) E(x, \xi) - F(x, \xi) u_k(x)] \, d\Gamma \} \end{aligned} \tag{46}$$

where

$$\begin{aligned} q(x) &= -[k(x) \nabla T(x)] \cdot \hat{n} \\ p_k(x) &= -[k(x) \nabla u_k(x)] \cdot \hat{n} \\ F(x, \xi) &= -[k(x) \nabla E(x, \xi)] \cdot \hat{n} \end{aligned} \tag{47}$$

Here, $\hat{n}$ is the outward-drawn normal to the boundary $\Gamma$. After discretization into $N$ boundary elements Eq. (46) can be expressed in matrix form as

$$\boldsymbol{G q} - \tilde{\boldsymbol{H}} \boldsymbol{T} = \sum_{k=1}^{N+L} \boldsymbol{\alpha}_k \{ \boldsymbol{G p}_k - \tilde{\boldsymbol{H}} \boldsymbol{u}_k \} \tag{48}$$

where $\tilde{H}$ is the modified influence coefficient matrix defined in Eq. (41). From Eq. (43), the vector $\boldsymbol{\alpha}_k$ can be expressed as

$$\rho c \frac{\partial T}{\partial t} = \sum_{k=1}^{N+L} \boldsymbol{\alpha}_k f_k \tag{49}$$

where the expansion functions satisfy

$$f_k = \nabla \cdot [k(x) \nabla u_k] \tag{50}$$

Collocating Eq. (49) at the $N + L$ dual reciprocity expansion points, leads to

$$\rho c \left( \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{t}} \right) = F \boldsymbol{\alpha} \tag{51}$$

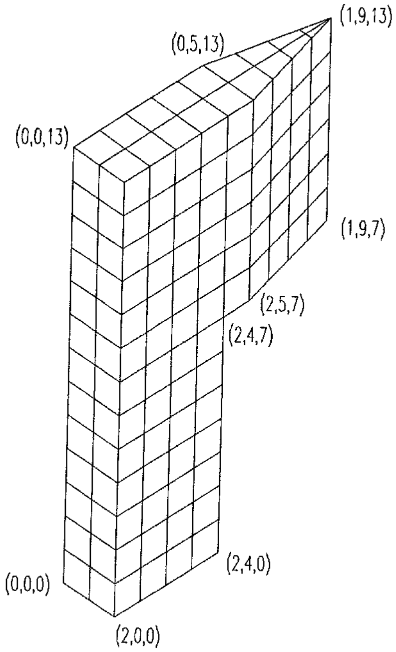


**Fig. 2 Geometry and BEM discretization of a thrust-vector-control vane.**

or

$$\boldsymbol{\alpha} = \rho c F^{-1}\left(\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{t}}\right) \tag{52}$$

Equation (52) into Eq. (48) leads to

$$\tilde{H}\boldsymbol{T} - G\boldsymbol{q} = C\left(\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{t}}\right) \tag{53}$$

where

$$C = -\rho c(GP - \tilde{H}U)F^{-1} \tag{54}$$

Here $C$ is a capacitance matrix, and the matrices $U$ and $P$ are obtained by evaluating the expansion functions $u_k$ and its normal derivatives $p_k$ at every dual reciprocity point, respectively. Applying a first-order finite difference in time

$$\tilde{H}\boldsymbol{T} - G\boldsymbol{q} = C\left(\frac{\boldsymbol{T}^{p+1} - \boldsymbol{T}^{p}}{\Delta t}\right) \tag{55}$$

where $p$ is the time-stepping parameter. Using Newmark parameters $\theta_t$ and $\theta_q$ to position the temperature vector $\boldsymbol{T}$ and the flux vector $\boldsymbol{q}$ between the time steps $p$ and $p + 1$, Eq. (55) becomes

$$(\Delta t\theta_t\tilde{H} - C)\boldsymbol{T}^{p+1} - (\Delta t\theta_q G)\boldsymbol{q}^{p+1} = [\Delta t(\theta_t - 1)\tilde{H} - C]\boldsymbol{T}^{p} + \Delta t(1 - \theta_q)G\boldsymbol{q}^{p} \tag{56}$$

where the RHS of Eq. (56) is known from the previous time step. The solution of Eq. (56) follows standard boundary element treatment for the computation of the influence coefficient matrices and the imposition of boundary conditions.

Finally, $u_k$ must be selected to satisfy the expressions derived earlier [Eq. (50)]. For the numerical implementation of the method, the most efficient and most commonly used form for the functions $u_k$ are based on radial distances,[20] in particular, for the three-dimensional case

$$u_k = \frac{r_k^2}{6}\left(\frac{r_k}{2} + 1\right) \tag{57}$$

where $r_k$ is the radial distance from every field point to the $k$th dual reciprocity point. Furthermore, the expressions for the functions $p_k$ and $f_k$ become

$$p_k(x) = -\left\{k(x)\nabla\left[\frac{r_k^2}{6}\left(\frac{r_k}{2} + 1\right)\right]\right\}\cdot\hat{n}$$
$$f_k(x) = \nabla\cdot\left\{k(x)\nabla\left[\frac{r_k^2}{6}\left(\frac{r_k}{2} + 1\right)\right]\right\} \tag{58}$$

It is noted that the new radial basis expansion functions used in this paper, as given in Eq. (58), retrieve the standard expansion functions in the case of constant conductivity.[24] In particular, in the case of constant thermal conductivity, Eq. (58) reduces to

$$f_k(x) = k(r_k + 1) \tag{59}$$

where $f_k$ is now the standard $(1 + r)$ type radial basis function.[20] Numerical examples are presented next to validate the preceding developments.

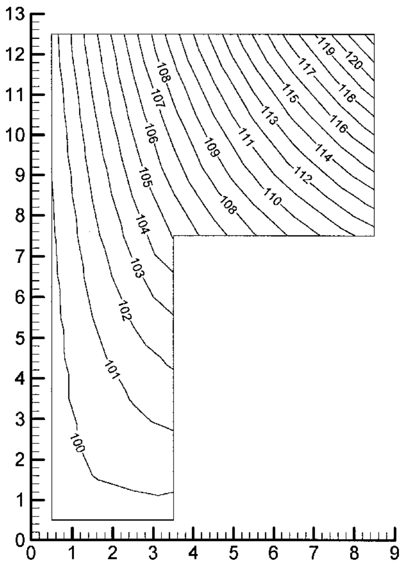


**Fig. 3 BEM computed isotherms.**

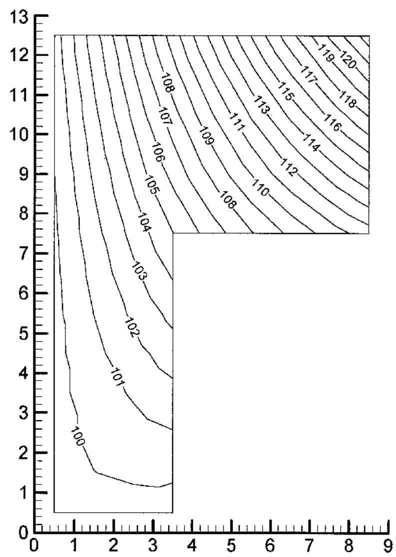


**Fig. 4 Exact isotherms distribution.**

## Numerical Examples

An extensive set of two-dimensional verification examples of the generalized boundary integral solution for steady-state heat conduction in heterogeneous media can be found in Kassab and Divo.[16–19] In this paper a steady-state, three-dimensional problem is presented to demonstrate the accuracy of the generalized BIE. A thrust-vector-control vane model is considered and discretized using 226 bilinear isoparametric boundary elements as shown in Fig. 2. The thermal conductivity is chosen to vary trilinearly as

$$k(x, y, z) = (2x + y + z + 20) \tag{60}$$

The following can be taken as an exact solution temperature distribution and can be used to impose boundary conditions on the vane geometry:

$$T(x, y, z) = 100 + 0.01(5x^2 - 5y^2 + z^2 - 9xy - 15xz + 26yz - 20x + 5y - 5z) \tag{61}$$

It can be readily verified by substitution of the preceding temperature distribution into Eq. (1) the thermal conductivity in Eq. (60) that the governing heat conduction equation is identically satisfied. Thus, using Eq. (61), first-kind boundary conditions are imposed on the vane, and the temperature distribution is computed using the generalized BIE, Eq. (7). Exact and BEM-computed isotherms are provided in Figs. 3 and 4. Examination of the relative percent error distribution at the mid-plane of the vane (Fig. 5), reveals a maximum error of approximately 0.34%. This validates our three-dimensional BEM code and further validates the generalized BIE for heterogeneous heat conduction [Eq. (7)].

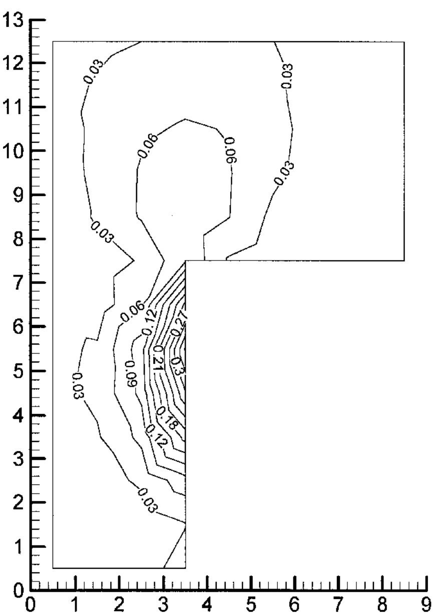


**Fig. 5 Percent relative error distribution.**

Next, the transient formulation is to be tested. In all cases we use fully implicit formulation $\theta_t = \theta_q = 1$. First, a unit cube is considered. It is discretized with 24 equally spaced bilinear boundary elements and nine internal dual reciprocity points. The initial temperature is set to a value of 100, and third-kind boundary conditions are imposed on every surface with a film coefficient of one and a convective temperature of zero. The problem is solved in a homogeneous medium ($k = 1$) to obtain an exact solution. The density $\rho$ and specific heat $c$ are set to one. Nine internal dual reciprocity points are used to obtain the BEM approximation. A comparison of the BEM and exact solutions at the center point of the cube as a function of time is provided in Fig. 6. It can be observed that the evolution of the two solutions in time follows the same trend with a maximum deviation of less then 3.97% between the two solutions.

Now that the transient formulation has been tested, a problem can be solved in a heterogeneous media, and verified against an asymptotic solution. Assume that the same geometry and discretization is used to solve for the temperature distribution in a medium with a spatially varying thermal conductivity, taken as

$$k(x, y, z) = (10 + x + 2y + 3z)^2 \tag{62}$$

The following asymptotic solution can be derived and used to impose the boundary conditions and to verify the BEM solution at large time

$$T(x, y, z) = \frac{44x + 50y + 66z}{10 + x + 2y + 3z} + 100 \tag{63}$$

Figure 7 provides a plot of the evolution of the BEM solution at the center point of the cube departing from an initial

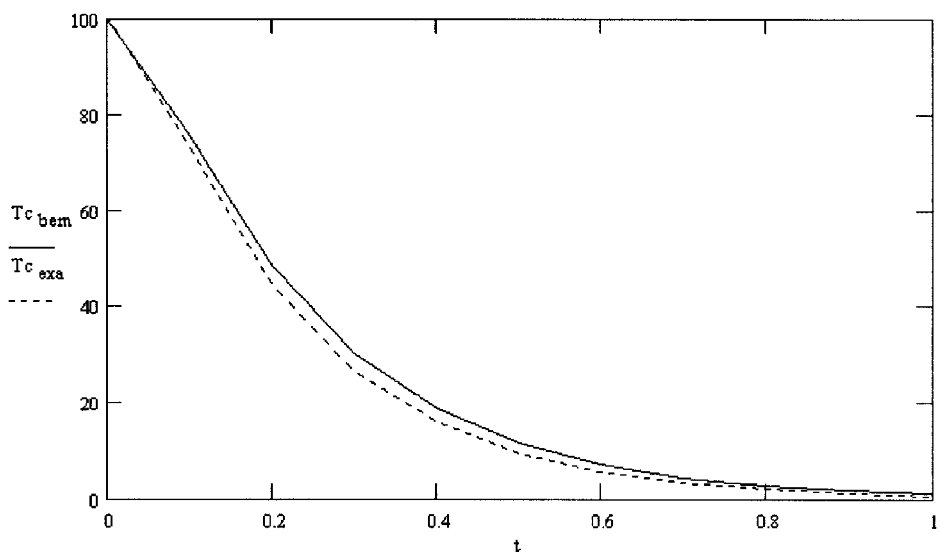


**Fig. 6 Comparison of the time evolution of the BEM and exact solutions at the center point (0.5, 0.5, 0.5).**

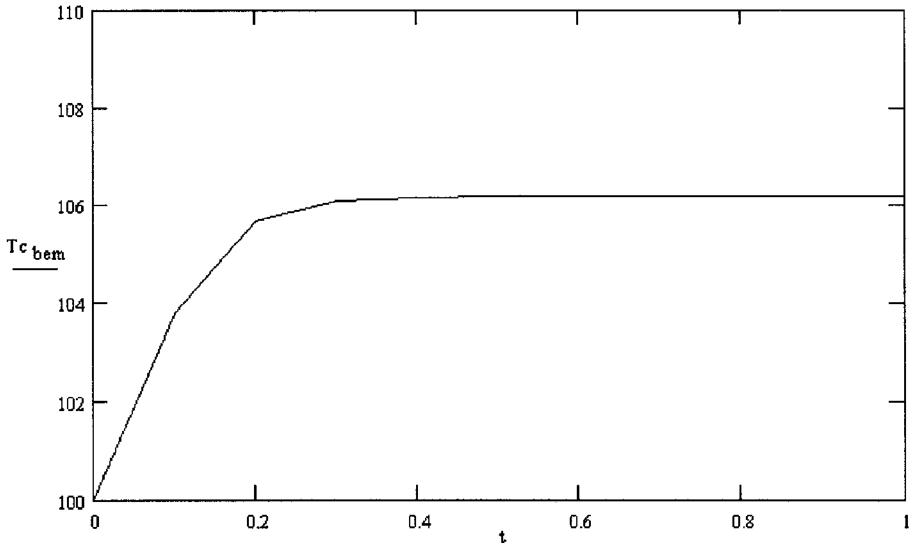


**Fig. 7 Time evolution of the BEM solution at the center point (0.5, 0.5, 0.5).**

**Table 1 BEM, exact solution, and relative percent error at the nine dual reciprocity points for large time**

| $x$ | $y$ | $z$ | $T_{bem}$ | $T_{exa}$ | Error % |
|---|---|---|---|---|---|
| 0.25 | 0.25 | 0.25 | 103.504 | 103.478 | 0.025 |
| 0.75 | 0.25 | 0.25 | 105.185 | 105.167 | 0.018 |
| 0.75 | 0.75 | 0.25 | 106.712 | 106.692 | 0.018 |
| 0.25 | 0.75 | 0.25 | 105.215 | 105.200 | 0.014 |
| 0.25 | 0.75 | 0.75 | 107.024 | 107.000 | 0.022 |
| 0.75 | 0.75 | 0.75 | 108.314 | 108.276 | 0.036 |
| 0.75 | 0.25 | 0.75 | 107.060 | 107.037 | 0.022 |
| 0.25 | 0.25 | 0.75 | 105.631 | 105.615 | 0.014 |
| 0.50 | 0.50 | 0.50 | 106.205 | 106.154 | 0.048 |

temperature of 100. The asymptotic values (at a very long time) of the computed BEM and exact temperatures, along with their relative percent error, are provided in Table 1 for the nine internal dual reciprocity points. Examination of the values in Table 1 reveal an excellent approximation of the asymptotic solution by the BEM transient formulation with a maximum error of 0.048%.

The final example is a two-dimensional transient problem in a NACA 0020 airfoil with a fictitious elliptic cooling duct (Fig. 8). An analytical solution is constructed to verify the transient BEM predictions. The temperature distribution is assumed to be

$$T(x, y, t) = \tan\left(\frac{x + y + 1}{2}\right) \exp\left(\frac{1}{\rho c} t\right) + 100 \qquad (64)$$

A spatially dependent thermal conductivity is then derived as

$$k(x, y) = 2\,\frac{\ell n\left[\sec\left(\frac{x + y + 1}{2}\right)\right]}{\sec^2\left(\frac{x + y + 1}{2}\right)} \qquad (65)$$

to satisfy the governing diffusion equation [Eq. (42)]. The temperature distribution in Eq. (64) is in turn used to impose the initial condition, first-kind boundary condition along the outer surface of the airfoil, and second-kind boundary condition on the surface of the cooling duct. An analytical solution is then available for comparison with numerical predictions.

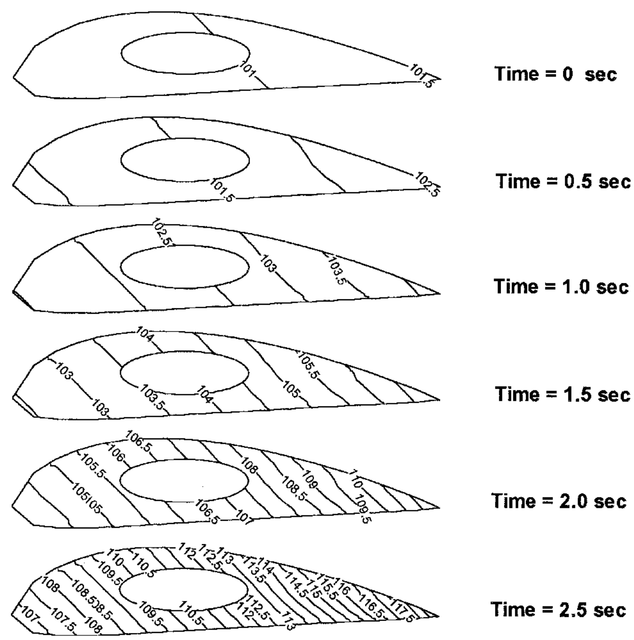


**Fig. 8 BEM temperature distribution from $t$ = 0–2.5.**

To obtain a closed-form expression for the generalized fundamental solution in Eq. (16), the thermal conductivity distribution is approximated by a least-squares fit of a complete biquadratic polynomial as

$$k_a(x + y) = 0.19728 + 0.38372x + 0.33572y - 0.22115x^2 - 0.31538xy - 0.14844y^2 \qquad (66)$$

Twenty quadratic boundary elements are used to discretize the outer boundary and another 20 are used for the inner boundary. Forty internal dual reciprocity points are employed in this example. The density and heat capacity are both set to one. Plots of the BEM-computed solution for $t$ = 0, 0.5, 1.0, 1.5, 2.0, and 2.5 are shown in Fig. 8. The exact temperature distribution according to Eq. (64) for the same six time steps is provided in Fig. 9, while the relative error percent distribution between the exact and the BEM solutions is provided in the set of plots of Fig. 10 for the same six time steps.

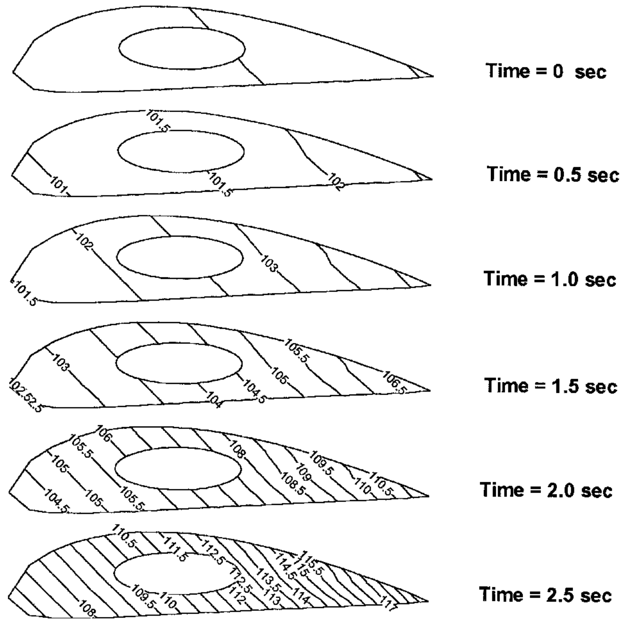


**Fig. 9 Exact temperature distribution from $t$ = 0–2.5.**

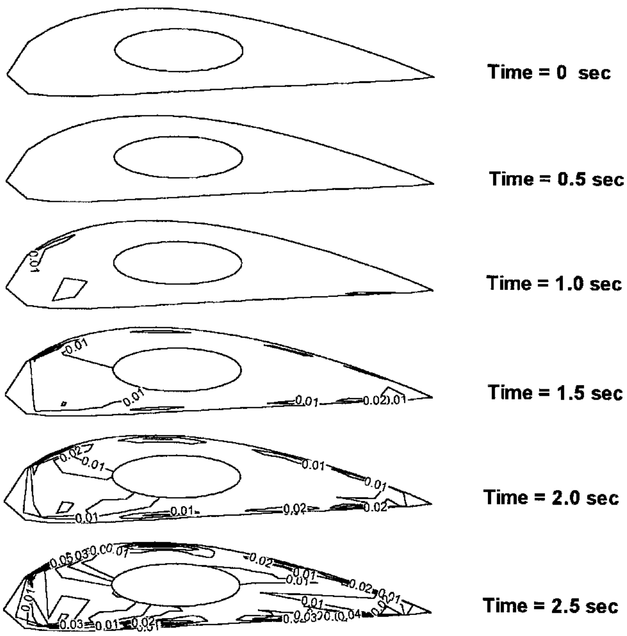


**Fig. 10 Relative error percent distribution from $t$ = 0–2.5.**

Excellent agreement between the two solutions is shown as the error distribution exhibits a maximum value of less than 0.06%. This completes the verification for the dual reciprocity extension of the generalized BIE developed to solve transient heat conduction in heterogeneous media.

## Conclusions

Using the generalized boundary integral method developed by the authors for steady heat conduction in heterogeneous media as a point of departure, a generalized dual reciprocity BEM is presented for the solution of transient heat conduction problems in heterogeneous media. In the process, new interpolating radial basis functions are defined. This method retains the boundary-only discretization feature of the BEM. Two- and three-dimensional numerical examples provided validation of the proposed method. The authors are currently researching an extension of the formulation for heat conduction in nonhomogeneous media to a direct time-dependent approach.

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